

Persistence of Activity in Threshold Contact Processes, an “Annealed Approximation” of Random Boolean Networks*

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ABSTRACT: We consider a model for gene regulatory networks that is a modification of Kauffman’s J Theor Biol 22 (1969), 437–467 random Boolean networks. There are three parameters: n = the number of nodes, r = the number of inputs to each node, and p = the expected fraction of 1’s in the Boolean functions at each node. Following a standard practice in the physics literature, we use a threshold contact process on a random graph on n nodes, in which each node has in degree r , to approximate its dynamics. We show that if $r \geq 3$ and $r \cdot 2p(1 - p) > 1$, then the threshold contact process persists for a long time, which correspond to chaotic behavior of the Boolean network. Unfortunately, we are only able to prove the persistence time is $\geq \exp(cn^{b(p)})$ with $b(p) > 0$ when $r \cdot 2p(1 - p) > 1$, and $b(p) = 1$ when $(r - 1) \cdot 2p(1 - p) > 1$. © 2011 Wiley Periodicals, Inc. Random Struct. Alg., 39, 228–246, 2011

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1. INTRODUCTION

Random Boolean networks were originally developed by Kauffman [9] as an abstraction of genetic regulatory networks. In our version of his model, the state of each node $x \in V_n \equiv \{1, 2, \dots, n\}$ at time $t = 0, 1, 2, \dots$ is $\eta_t(x) \in \{0, 1\}$, and each node x receives input from r distinct nodes $y_1(x), \dots, y_r(x)$, which are chosen randomly from $V_n \setminus \{x\}$.

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We construct our random directed graph G_n on the vertex set $V_n = \{1, 2, \dots, n\}$ by putting oriented edges to each node from its input nodes. To be precise, we define the graph by creating a random mapping $\phi : V_n \times \{1, 2, \dots, r\} \rightarrow V_n$, where $\phi(x, i) = y_i(x)$, such that $y_i(x) \neq x$ for $1 \leq i \leq r$ and $y_i(x) \neq y_j(x)$ when $i \neq j$, and taking the edge set $E_n \equiv \{(y_i(x), x) : 1 \leq i \leq r, x \in V_n\}$. So each vertex has in-degree r in our random graph G_n . The total number of choices for ϕ is $[(n-1)(n-2) \cdots (n-r)]^n$. However, the resulting graph G_n will remain the same under any permutation of the vector $\mathbf{y}_x \equiv (y_1(x), \dots, y_r(x))$ for any $x \in V_n$. So if $e_{zx} \in \{0, 1\}$ is the number of directed edges from node z to node x in G_n , then $\sum_{z=1}^n e_{zx} = r$, and the total number of permutations of the vectors \mathbf{y}_x , $1 \leq x \leq n$, that correspond to the same graph is $(r!)^n$. So if \mathbb{P} denotes the distribution of G_n , then

$$\mathbb{P}(e_{zx}, 1 \leq z, x \leq n) = \frac{(r!)^n}{[(n-1)(n-2) \cdots (n-r)]^n} = \frac{1}{\left[\binom{n-1}{r}\right]^n},$$

if $e_{zx} \in \{0, 1\}$, $e_{xx} = 0$ and $\sum_{z=1}^n e_{zx} = r$ for all $x \in V_n$, and $\mathbb{P}(e_{zx}, 1 \leq x, z \leq n) = 0$ otherwise. So our random graph G_n has uniform distribution over the collection of all directed graphs on the vertex set V_n in which each vertex has in-degree r . Once chosen the network remains fixed through time. The rule for updating node x is

$$\eta_{t+1}(x) = f_x(\eta_t(y_1(x)), \dots, \eta_t(y_r(x))),$$

where the values $f_x(v)$, $x \in V_n$, $v \in \{0, 1\}^r$, chosen at the beginning and then fixed for all time, are independent and $= 1$ with probability p .

A number of simulation studies have investigated the behavior of this model. See Kadanoff et al. [8] for survey. Flyvberg and Kjaer [6] have studied the degenerate case of $r = 1$ in detail. Derrida and Pomeau [2] have argued that for $r \geq 3$ there is a phase transition in the behavior of these networks between rapid convergence to a fixed point and exponentially long persistence of changes, and identified the phase transition curve to be given by the equation $r \cdot 2p(1-p) = 1$. The networks with parameters below the curve have behavior that is “ordered”, and those with parameters above the curve have “chaotic” behavior. Since chaos is not healthy for a biological network, it should not be surprising that real biological networks avoid this phase. See Kauffman [10], Shmulevich et al. [13], and Nykter et al. [12].

To explain the intuition behind the conclusion of Derrida and Pomeau [2], we define another process $\{\zeta_t(x) : t \geq 1\}$ for $x \in V_n$, which they called the *annealed approximation*. The idea is that $\zeta_t(x) = 1$ if and only if $\eta_t(x) \neq \eta_{t-1}(x)$, and $\zeta_t(x) = 0$ otherwise. Now if the state of at least one of the inputs $y_1(x), \dots, y_r(x)$ into node x has changed at time t , then the state of node x at time $t+1$ will be computed by looking at a different value of f_x . If we ignore the fact that we may have used this entry before, we get the dynamics of the threshold contact process

$$\begin{aligned} P(\zeta_{t+1}(x) = 1 | \zeta_t(y_1(x)) + \cdots + \zeta_t(y_r(x)) > 0) &= 2p(1-p), \text{ and} \\ P(\zeta_{t+1}(x) = 0 | \zeta_t(y_1(x)) + \cdots + \zeta_t(y_r(x)) = 0) &= 1. \end{aligned}$$

Conditional on the state at time t , the decisions on the values of $\zeta_{t+1}(x)$, $x \in V_n$, are made independently.

We content ourselves to work with the threshold contact process, since it gives an approximate sense of the original model, and we can prove rigorous results about its behavior. To simplify notation and explore the full range of threshold contact processes we let

$q \equiv 2p(1-p)$, and suppose $0 \leq q \leq 1$. As mentioned above, it is widely accepted that the condition for prolonged persistence of the threshold contact process is $qr > 1$. To explain this, we note that vertices in the graph G_n have average out-degree r , so a value of 1 at a vertex will, on the average, produce qr 1's in the next generation.

We will also write the threshold contact process as a set valued process. Let $\xi_t \equiv \{x : \zeta_t(x) = 1\}$. We will refer to the vertices $x \in \xi_t$ as occupied at time t . So if P_{G_n} is the distribution of the threshold contact process $\xi \equiv \{\xi_t : t \geq 0\}$ conditioned on the graph G_n , then

$$P_{G_n}(x \in \xi_{t+1} | \{y_1(x), \dots, y_r(x)\} \cap \xi_t \neq \emptyset) = q, \text{ and} \\ P_{G_n}(x \in \xi_{t+1} | \{y_1(x), \dots, y_r(x)\} \cap \xi_t = \emptyset) = 0,$$

and if \mathbf{P} denotes the distribution of the threshold contact process on the random graph G_n , which has distribution \mathbb{P} , then

$$\mathbf{P}(\cdot) = \mathbb{E}P_{G_n}(\cdot), \quad (1.1)$$

where \mathbb{E} is the expectation corresponding to the probability distribution \mathbb{P} .

Let $\xi^A \equiv \{\xi_t^A : t \geq 0\}$ denote the threshold contact process starting from $\xi_0^A = A \subset V_n$, and $\xi^1 \equiv \{\xi_t^1 : t \geq 0\}$ denote the special case when $A = V_n$. Let ρ be the survival probability of a branching process with offspring distribution $p_r = q$ and $p_0 = 1 - q$. By branching process theory

$$\rho = 1 - \theta, \text{ where } \theta \in (0, 1) \text{ satisfies } \theta = 1 - q + q\theta^r. \quad (1.2)$$

Using all the ingredients above we now present our first result.

Theorem 1. *Suppose $q(r-1) > 1$ and let $\delta > 0$. Let \mathbf{P} be the probability distribution in (1.1). Then there is a positive constant $C(\delta)$ so that as $n \rightarrow \infty$*

$$\inf_{t \leq \exp(C(\delta)n)} \mathbf{P}\left(\frac{|\xi_t^1|}{n} \geq \rho - 2\delta\right) \rightarrow 1.$$

The threshold contact process will eventually die out on any finite graph. But it certainly cannot last longer than $\exp(O(n))$ units of time, because the number of vertices is n , and so even if all vertices are occupied at time 0, there is a probability $\geq (1-q)^n$ that all of them will be vacant at time 1.

To prove Theorem 1, we will consider the dual coalescing branching process $\hat{\xi} \equiv \{\hat{\xi}_t : t \geq 0\}$. In this process if x is occupied at time t , then with probability q all of the sites $y_1(x), \dots, y_r(x)$ will be occupied at time $t+1$, and with probability $1-q$ none of them will be occupied at time $t+1$. Birth events from different sites are independent. Let $\hat{\xi}^A \equiv \{\hat{\xi}_t^A : t \geq 0\}$ be the dual process starting from $\hat{\xi}_0^A = A \subset V_n$. The two processes can be constructed on the same sample space so that for any choices of A and B for the initial sets of occupied sites, ξ^A and $\hat{\xi}^B$ satisfies the following duality relationship, see Griffeath [7].

$$\{\xi_t^A \cap B \neq \emptyset\} = \{\hat{\xi}_t^B \cap A \neq \emptyset\}, \quad t = 0, 1, 2, \dots \quad (1.3)$$

Taking $A = \{1, 2, \dots, n\}$ and $B = \{x\}$ this says

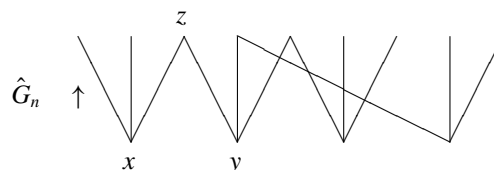
$$\{x \in \xi_t^1\} = \{\hat{\xi}_t^{\{x\}} \neq \emptyset\}, \quad (1.4)$$

From (1.3) it should be clear that we can prove Theorem 1 by studying the coalescing branching process. The key to this is an “isoperimetric inequality”. Let \bar{G}_n be the graph obtained from our original graph $G_n = (V_n, E_n)$ by reversing the edges. That is, $\bar{G}_n = (V_n, \bar{E}_n)$, where $\bar{E}_n = \{(x, y) : (y, x) \in E_n\}$. Given a set $U \subset V_n$, let

where $x \rightarrow y$ means $(x, y) \in \hat{E}_n$. Note that U^* can contain vertices of U . The idea behind this definition is that if U is occupied at time t in the coalescing branching process, then the vertices in U^* may be occupied at time $t + 1$.

$$\mathbb{P}[E(m, (r-1-\eta)m)] \leq \exp(-\eta m \log(n/m)/2).$$

Sketch of Proof. Define an undirected graph H_n on the vertex set V_n so that x and y are adjacent in H_n if and only if there is a z so that $x \rightarrow z$ and $y \rightarrow z$ in \hat{G}_n . The drawing illustrates the case $r = 3$.



Since the isoperimetric constant is $\leq r - 1$, it follows that when $q(r - 1) < 1$, then for any $\epsilon > 0$ there are bad sets A with $|A| \leq n\epsilon$, so that $E|\hat{\xi}_1^A| \leq |A|$. Computations from the proof of Theorem 2 suggest that there are a large number of bad sets. We have no idea how to bound the amount of time spent in bad sets, so we have to take a different approach to show persistence when $1/r < q \leq 1/(r - 1)$.

Theorem 3. Suppose $qr > 1$. If δ_0 is small enough, then for any $0 < \delta < \delta_0$, there are constants $C(\delta) > 0$ and $B(\delta) = (1/8 - 2\delta) \log(qr - \delta) / \log r$ so that as $n \rightarrow \infty$

$$\inf_{t \leq \exp(C(\delta) \cdot n^{B(\delta)})} \mathbf{P} \left(\frac{|\xi_t^1|}{n} \geq \rho - 2\delta \right) \rightarrow 1.$$

Based on results for the basic contact process on $(\mathbf{Z} \bmod n)$ by Durrett and Liu [4] and Durrett and Schonmann [5], and on $(\mathbf{Z} \bmod n)^d$ by Mountford [11], it is natural to believe that the conclusion of Theorem 1 holds in all situations with $qr > 1$. But here we content ourselves with the rather weak result.

To prove Theorem 3, we will again investigate persistence of the dual. Let

$$\begin{aligned} d_0(x, y) &\equiv \text{length of a shortest oriented path from } x \text{ to } y \text{ in } \hat{G}_n, \\ d(x, y) &\equiv \min_{z \in V_n} [d_0(x, z) + d_0(y, z)], \end{aligned} \quad (1.6)$$

and for any subset A of vertices let

$$m(A, K) = \max_{S \subseteq A} \{|S| : d(x, y) \geq K \text{ for all } x, y \in S, x \neq y\}. \quad (1.7)$$

Let $R \equiv \log n / \log r$ be the average value of $d_0(1, 2)$. So R is an average distance between any two distinct vertices of the graph. Also let $a = 1/8 - \delta$ and $B = (a - \delta) \log(qr - \delta) / \log r$. We will show that if $m(\hat{\xi}_s^A, 2\lceil aR \rceil) < \lfloor n^B \rfloor$ at some time s , then with high probability, we will later have $m(\hat{\xi}_t^A, 2\lceil aR \rceil) \geq \lfloor n^B \rfloor$ for some $t > s$. To do this we explore the vertices in \hat{G}_n one at a time using a breadth-first search algorithm based on the distance function d_0 . We say that a collision has occurred if we encounter a vertex more than once in the exploration process. First we show in Lemma 3.1 that, with probability tending to 1 as $n \rightarrow \infty$, there can be at most one collision in the set $\{u : d_0(x, u) \leq 2\lceil aR \rceil\}$ for any $x \in V_n$. Then we argue in Lemma 3.2 that when we first have $m(\hat{\xi}_s^A, 2\lceil aR \rceil) < \lfloor n^B \rfloor$, there is a subset N of occupied sites so that $|N| \geq (q - \delta) \lfloor n^B \rfloor$, and $d(z, w) \geq 2\lceil aR \rceil - 2$ for any two distinct vertices $z, w \in N$, and $\{u : d_0(z, u) \leq 2\lceil aR \rceil - 1\}$ has no collision. We run the dual process starting from the vertices of N until time $\lceil aR \rceil - 1$, so they are independent. With high probability there will be at least one vertex $w \in N$ for which $|\hat{\xi}_{\lceil aR \rceil - 1}^{(w)}| \geq \lceil n^B \rceil$. By the choice of N , for any two distinct vertices $x, z \in \hat{\xi}_{\lceil aR \rceil - 1}^{(w)}$, $d(x, z) \geq 2\lceil aR \rceil$. It seems foolish to pick only one vertex w , but we do not know how to guarantee that the vertices are suitably separated if we pick more.

2. PROOF OF THEOREM 1

We begin with the proof of the isoperimetric inequality, Theorem 2.

Proof of Theorem 2. Let $p(m, k)$ be the probability that there is a set U with $|U| = m$ and $|U^*| = k$. First we will estimate $p(m, \ell)$ where $\ell = \lfloor (r - 1 - \eta)m \rfloor$.

$$p(m, \ell) \leq \sum_{\{(U, U') : |U| = m, |U'| = \ell\}} \mathbb{P}(U^* = U') \leq \sum_{\{(U, U') : |U| = m, |U'| = \ell\}} \mathbb{P}(U^* \subset U').$$

According to the construction of G_n , for any $x \in U$ the other ends of the r edges coming out of it are distinct and they are chosen at random from $V_n \setminus \{x\}$. So

$$\mathbb{P}(U^* \subset U') = \left[\frac{\binom{|U'|}{r}}{\binom{n-1}{r}} \right]^{|U|} \leq \left(\frac{|U'|}{n-1} \right)^{r|U|},$$

and hence

$$p(m, \ell) \leq \binom{n}{m} \binom{n}{\ell} \left(\frac{\ell}{n-1} \right)^{rm}. \quad (2.1)$$

To bound the right-hand side, we use the trivial bound

$$\binom{n}{m} \leq \frac{n^m}{m!} \leq \left(\frac{ne}{m} \right)^m, \quad (2.2)$$

where the second inequality follows from $e^m > m^m/m!$. Using (2.2) in (2.1)

$$p(m, \ell) \leq (ne/m)^m (ne/\ell)^\ell \left(\frac{\ell}{n} \right)^{rm} \left(\frac{n}{n-1} \right)^{rm}.$$

Recalling $\ell \leq (r-1-\eta)m$, and accumulating the terms involving (m/n) , $r-1-\eta$ and e the last expression becomes

$$\begin{aligned} &\leq e^{m(r-\eta)} (m/n)^{m[-1-(r-1-\eta)+r]} (r-1-\eta)^{-(r-1-\eta)m+rm} [n/(n-1)]^{rm} \\ &= e^{m(r-\eta)} (m/n)^{m\eta} (r-1-\eta)^{m(1+\eta)} [n/(n-1)]^{rm}. \end{aligned}$$

Letting $c(\eta) = r - \eta + r \log(n/(n-1)) + (1+\eta) \log(r-1-\eta) \leq C$ for $\eta \in (0, r-1)$, we have

$$p(m, \lfloor (r-1-\eta)m \rfloor) \leq \exp(-\eta m \log(n/m) + Cm).$$

Summing over integers $k = (r-1-\eta')m$ with $\eta' \geq \eta$, and noting that there are fewer than rm terms in the sum, we have

$$\mathbb{P}[E(m, (r-1-\eta)m)] \leq \exp(-\eta m \log(n/m) + C'm).$$

To clean up the result to the one given in Theorem 2, choose ϵ_0 such that $\eta \log(1/\epsilon_0)/2 > C'$. Hence for any $m \leq \epsilon_0 n$,

$$\eta \log(n/m)/2 \geq \eta \log(1/\epsilon_0)/2 > C',$$

which gives the desired result. ■

Our next goal is to show that the graph \hat{G}_n locally looks like a tree with high probability. For that we explore all the vertices in V_n one at a time, starting from a vertex x , and using a breadth-first search algorithm based on the distance function d_0 of (1.6). More precisely, for each $x \in V_n$, we define the sets A_x^k , which we call the active set at the k^{th} step, and R_x^k , which we call the removed set at k^{th} step, for $k = 0, 1, \dots, \beta_x$, where $\beta_x \equiv \min\{l : A_x^l = \emptyset\}$, sequentially as follows. $R_x^0 \equiv \emptyset$ and $A_x^0 \equiv \{x\}$. Let $D(x, l) = \{y : d_0(x, y) \leq l\}$. For

$0 \leq k < \beta_x$, we get $k_0 = \min\{l : 0 \leq l \leq k, A_x^l \cap D(x, l) \neq \emptyset\}$, and choose $x_k \in A_x^k \cap D(x, k_0)$ with the minimum index.

$$\begin{aligned} \text{If } x_k \in R_x^k, & \quad \text{then } A_x^{k+1} \equiv A_x^k \setminus \{x_k\}, R_x^{k+1} \equiv R_x^k \text{ and} \\ \text{if } x_k \notin R_x^k, & \quad \text{then } A_x^{k+1} \equiv A_x^k \cup \{y_1(x_k), \dots, y_r(x_k)\} \setminus \{x_k\}, R_x^{k+1} \equiv R_x^k \cup \{x_k\}. \end{aligned}$$

If $x_k \in R_x^k$, we say that a collision has occurred while exploring \hat{G}_n starting from x . The choice of x_k ensures that while exploring the graph starting from x , for any $j \geq 1$, we consider the vertices, which are at d_0 distance j from x , prior to those, which are at d_0 distance $j + 1$ from x .

The next Lemma shows that with high probability R_x^k will have k vertices, and for $x \neq z$, R_x^k and R_z^k do not intersect each other, when $k \leq n^{1/2-\delta}$. For the lemma we need the following stopping times.

$$\begin{aligned} \pi_x^1 &\equiv \min\{l \geq 1 : |R_x^l| < l\}, \\ \pi_{x,z} &\equiv \min\{l \geq 1 : R_x^l \cap R_z^l \neq \emptyset\}, x \neq z, \\ \alpha_x^{n,\delta} &\equiv \min\{l \geq 1 : |R_x^l| \geq \lceil n^{1/2-\delta} \rceil\}, \delta < 1/2, \\ \beta_x &= \min\{l \geq 1 : A_x^l = \emptyset\} \end{aligned} \quad (2.3)$$

So π_x^1 is the time of first collision while exploring \hat{G}_n starting from x , and $\pi_{x,z}$ is the time of first collision while exploring \hat{G}_n simultaneously from x and z .

Lemma 2.1. Suppose $0 < \delta < 1/2$. Let $I_x^1, x \in V_n$, and $I_{x,z}, x, z \in V_n, x \neq z$, be the events

$$I_x^1 \equiv \{\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,\delta}\}, \quad I_{x,z} \equiv I_x^1 \cap I_z^1 \cap \{\pi_{x,z} \geq \alpha_x^{n,\delta} \vee \alpha_z^{n,\delta}\},$$

where $\pi_x^1, \pi_{x,z}, \alpha_x^{n,\delta}$ and β_x are the stopping times defined in (2.3). Then

$$\mathbb{P}[(I_x^1)^c] \leq n^{-2\delta}, \quad \mathbb{P}(I_{x,z}^c) \leq 5n^{-2\delta} \quad (2.4)$$

for large enough n .

Note that the randomness, which determines whether the events I_x^1 and $I_{x,z}$ occur or not, arises only from the construction of the random graph G_n , and does not involve the threshold contact process ξ^1 on G_n .

Proof. Let $\delta' = 1/2 - \delta$. Since in the construction of the random graph G_n the input nodes $y_i(z), 1 \leq i \leq r$, for any vertex z are distinct and different from z , there are at least $n - r$ choices for each $y_i(z)$. Also $|R_x^l| \leq l$ for any l . So

$$\mathbb{P}(|R_x^k| = |R_x^{k-1}|) \leq (k-1)/(n-r). \quad (2.5)$$

It is easy to check that $\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,\delta}$ if $|R_x^k| \neq |R_x^{k-1}|$ for $k = 1, 2, \dots, \lceil n^{\delta'} \rceil$. So

$$\begin{aligned} \mathbb{P}[(I_x^1)^c] &\leq \mathbb{P}[\cup_{k=1}^{\lceil n^{\delta'} \rceil} (|R_x^k| = |R_x^{k-1}|)] \leq \sum_{k=1}^{\lceil n^{\delta'} \rceil} \mathbb{P}(|R_x^k| = |R_x^{k-1}|) \\ &\leq \sum_{k=1}^{\lceil n^{\delta'} \rceil} (k-1)/(n-r) \leq n^{2\delta'}/n = n^{-2\delta} \end{aligned}$$

for large enough n . For the other assertion, note that $I_{x,z}$ occurs if $|R_x^k| \neq |R_x^{k-1}|$, $|R_z^k| \neq |R_z^{k-1}|$ and $R_x^k \cap R_z^k = \emptyset$ for $k = 1, 2, \dots, \lceil n^{\delta'} \rceil$. Also if for some $k \geq 1$ $R_x^k \cap R_z^k \neq \emptyset$ and $R_x^l \cap R_z^l = \emptyset$ for all $1 \leq l < k$, then either $R_x^k = R_x^{k-1} \cup \{x_{k-1}\}$ and $x_{k-1} \in R_z^{k-1}$, or $R_z^k = R_z^{k-1} \cup \{z_{k-1}\}$ and $z_{k-1} \in R_x^k$. Now since each of the input nodes in the construction of G_n has at least $n - r$ choices, and $|R_x^l|, |R_z^l| \leq l$ for any l ,

$$\begin{aligned} \mathbb{P}(R_x^k \cap R_z^k \neq \emptyset, R_x^l \cap R_z^l = \emptyset, 1 \leq l < k) \\ \leq \mathbb{P}(x_{k-1} \in R_z^{k-1}) + \mathbb{P}(z_{k-1} \in R_x^k) \leq (2k-1)/(n-r). \end{aligned} \quad (2.6)$$

Combining the error probabilities of (2.5) and (2.6)

$$\begin{aligned} \mathbb{P}(I_{x,z}^c) &\leq \mathbb{P}\left[\bigcup_{k=1}^{\lceil n^{\delta'} \rceil} (|R_x^k| = |R_x^{k-1}|) \cup \bigcup_{k=1}^{\lceil n^{\delta'} \rceil} (|R_z^k| = |R_z^{k-1}|) \cup \bigcup_{k=1}^{\lceil n^{\delta'} \rceil} (R_x^k \cap R_z^k \neq \emptyset)\right] \\ &\leq \sum_{k=1}^{\lceil n^{\delta'} \rceil} [\mathbb{P}(|R_x^k| = |R_x^{k-1}|) + \mathbb{P}(|R_z^k| = |R_z^{k-1}|) + \mathbb{P}(R_x^k \cap R_z^k \neq \emptyset, R_x^l \cap R_z^l = \emptyset, 1 \leq l < k)] \\ &\leq \sum_{k=1}^{\lceil n^{\delta'} \rceil} (4k-3)/(n-r) \leq 5n^{2\delta'-1} = 5n^{-2\delta} \end{aligned}$$

for large n . ■

Lemma 2.1 shows that \hat{G}_n is locally tree-like. The number of vertices in the induced subgraph $\hat{G}_{x,M}$ with vertex set $G_n \cap \{u : d_0(x, u) \leq M\}$ is at most $1 + r + \dots + r^M \leq 2r^M$. So if I_x^1 occurs, then, for any M satisfying $2r^M \leq n^{1/2-\delta}$, the subgraph $\hat{G}_{x,M}$ is an oriented finite r -tree, where each vertex except the leaves has out-degree r . Similarly if $I_{x,z}$ occurs, then for any such M , $\hat{G}_{x,M} \cap \hat{G}_{z,M} = \emptyset$.

In the next lemma, we will use this to get a bound on the survival of the dual process for small times. Let ρ be the branching process survival probability defined in (1.2).

Lemma 2.2. *If $q > 1/r$, $\delta \in (0, qr - 1)$, $\gamma = (20 \log r)^{-1}$, and $b = \gamma \log(qr - \delta)$ then for any $x \in V_n$, if n is large,*

$$\mathbf{P}(|\hat{\xi}_{\lceil 2\gamma \log n \rceil}^{(x)}| \geq \lceil n^b \rceil) \geq \rho - \delta.$$

Proof. Let I_x^1 be the event

$$I_x^1 = \{\pi_x^1 \wedge \beta_x \geq \alpha_x^{n,1/4}\},$$

where $\pi_x^1, \beta_x, \alpha_x^{n,1/4}$ are as in (2.3). Let P_{Z^x} be the distribution of a branching process $\mathbf{Z}^x \equiv \{Z_t^x : t = 0, 1, 2, \dots\}$ with $Z_0^x = 1$ and offspring distribution $p_0 = 1 - q$ and $p_r = q$. Since $q > 1/r$, this is a supercritical branching process. Let B_x be the event that the branching process survives. Then

$$P_{Z^x}(B_x) = \rho,$$

where ρ is as in (1.2). If we condition on B_x , then, using a large deviation result for branching processes from Athreya [1],

$$P_{Z^x}\left(\left|\frac{Z_{t+1}^x}{Z_t^x} - qr\right| > \delta \mid B_x\right) \leq e^{-c(\delta)t} \quad (2.7)$$

for some constant $c(\delta) > 0$ and for large enough t . So if $F_x = \{Z_{t+1}^x \geq (qr - \delta)Z_t^x\}$ for $\lfloor \gamma \log n \rfloor \leq t < \lceil 2\gamma \log n \rceil$, then

$$P_{Z^x}(F_x^c | B_x) \leq \sum_{t=\lfloor \gamma \log n \rfloor}^{\lceil 2\gamma \log n \rceil - 1} e^{-c(\delta)t} \leq C_\delta n^{-c(\delta)\gamma/2} \quad (2.8)$$

for some constant $C_\delta > 0$ and for large enough n . On the event $B_x \cap F_x$,

$$Z_{\lceil 2\gamma \log n \rceil}^x \geq (qr - \delta)^{\lceil 2\gamma \log n \rceil - \lfloor \gamma \log n \rfloor} \geq (qr - \delta)^{\gamma \log n} = n^{\gamma \log(qr - \delta)},$$

since $Z_{\lfloor \gamma \log n \rfloor}^x \geq 1$ on B_x .

Now coming back to the dual process $\hat{\xi}^{\{x\}}$, let $P_{I_x^1}$ denote the conditional distribution of $\hat{\xi}^{\{x\}}$ given I_x^1 . This does not specify the entire graph but we will only use the conditional law for events that involve the process on the subtree whose existence is guaranteed by I_x^1 . By the choice of γ , the number of vertices in the subgraph induced by $\{u : d_0(x, u) \leq \lceil 2\gamma \log n \rceil\}$ is at most $2r^{\lceil 2\gamma \log n \rceil} < n^{1/4}$. Then it is easy to see that we can couple $P_{I_x^1}$ with P_{Z^x} so that

$$P_{I_x^1}[(|\hat{\xi}_t^{\{x\}}|, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot] = P_{Z^x}[(Z_t^x, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot].$$

Combining the error probabilities of (2.4) and (2.8)

$$\begin{aligned} \mathbf{P}(|\hat{\xi}_{\lceil 2\gamma \log n \rceil}^{\{x\}}| \geq \lceil n^b \rceil) &\geq P_{I_x^1}(|\hat{\xi}_{\lceil 2\gamma \log n \rceil}^{\{x\}}| \geq \lceil n^b \rceil) \mathbb{P}(I_x^1) \\ &= P_{Z^x}(Z_{\lceil 2\gamma \log n \rceil}^x \geq \lceil n^b \rceil) \mathbb{P}(I_x^1) \\ &\geq P_{Z^x}(B_x \cap F_x) \mathbb{P}(I_x^1) \\ &= P_{Z^x}(B_x) P_{Z^x}(F_x | B_x) \mathbb{P}(I_x^1) \\ &\geq \rho(1 - C_\delta n^{-c(\delta)\gamma/2})(1 - n^{-1/2}) \geq \rho - \delta \end{aligned}$$

for large enough n . ■

Lemma 2.2 shows that the dual process starting from one vertex will with probability $\geq \rho - \delta$ survive until there are $\lceil n^b \rceil$ many occupied sites. The next lemma will show that if the dual starts with $\lceil n^b \rceil$ many occupied sites, then for some $\epsilon > 0$ it will have $\lceil \epsilon n \rceil$ many occupied sites within time $\lceil \epsilon n \rceil$ with high probability.

Lemma 2.3. *If $q(r - 1) > 1$, then there exists $\epsilon_1 > 0$ such that for any A with $|A| \geq \lceil n^b \rceil$ the dual process $\hat{\xi}^A$ satisfies*

$$\mathbf{P}\left(\max_{t \leq \lceil \epsilon_1 n - n^b \rceil} |\hat{\xi}_t^A| < \epsilon_1 n\right) \leq \exp(-n^{b/4}).$$

Proof. Choose $\eta > 0$ such that $(q - \eta)(r - 1 - \eta) > 1$, and let $\epsilon_0(\eta)$ be the constant in Theorem 2. Take $\epsilon_1 \equiv \epsilon_0(\eta)$. Let $\nu \equiv \min\{t : |\hat{\xi}_t^A| \geq \lceil \epsilon_1 n \rceil\}$. Let $F_t \equiv \{|\hat{\xi}_t^A| \geq |\hat{\xi}_{t-1}^A| + 1\}$, and

$$\begin{aligned} B_t &\equiv \{\text{at least } (q - \eta)|\hat{\xi}_t^A| \text{ occupied sites of } \hat{\xi}_t^A \text{ give birth}\}, \\ C_t &\equiv \{|U_t^*| \geq (r - 1 - \eta)|U_t|\}, \text{ where } U_t = \{x \in \hat{\xi}_t^A : x \text{ gives birth}\}. \end{aligned}$$

Now if B_t and C_t occur, then

$$|\hat{\xi}_{t+1}^A| = |U_t^*| \geq (r-1-\eta)|U_t| \geq (r-1-\eta)(q-\eta)|\hat{\xi}_t^A| > |\hat{\xi}_t^A|, \quad (2.9)$$

i.e. F_{t+1} occurs. So $F_{t+1} \supseteq B_t \cap C_t$ for all $t \geq 0$. Using the binomial large deviations, see Lemma 2.3.3 on page 40 in Durrett [3],

$$P_{G_n}(B_t | \hat{\xi}_t^A) \geq 1 - \exp(-\Gamma((q-\eta)/q)q|\hat{\xi}_t^A|), \quad (2.10)$$

where $\Gamma(x) = x \log x - x + 1 > 0$ for $x \neq 1$. If we take $H_0 \equiv \{|\hat{\xi}_0^A| \geq \lceil n^b \rceil\}$ and $H_t \equiv \bigcap_{s=1}^t F_s$, then $|\hat{\xi}_t^A| \geq \lceil n^b \rceil$ on the event H_t for all $t \geq 0$. Keeping that in mind we can replace $|\hat{\xi}_t^A|$ in the right side of (2.10) by n^b to have

$$P_{G_n}(B_t^c \cap H_t) \leq P_{G_n}(B_t^c \cap \{|\hat{\xi}_t^A| \geq \lceil n^b \rceil\}) \leq \exp(-\Gamma((q-\eta)/q)qn^b) \quad \forall t \geq 0. \quad (2.11)$$

The same bound also works for the unconditional probability distribution \mathbf{P} . Next we see that $P_{G_n}(C_t | U_t) \geq \mathbf{1}_{E^c}$, where $E = E(|U_t|, (r-1-\eta)|U_t|)$, as defined in Theorem 2. Taking expectation with respect to the distribution of G_n , $\mathbf{P}(C_t | U_t) \geq \mathbb{P}(E^c)$. Since for $t < \nu$, $|U_t| < \epsilon_0(\eta)n$, and $|U_t| \geq (q-\eta)n^b \geq n^b/(r-1)$ on $H_t \cap B_t$, using Theorem 2

$$\begin{aligned} \mathbf{P}(C_t^c \cap B_t \cap H_t \cap \{t < \nu\}) &\leq \mathbf{P}[C_t^c \cap \{(n^b/(r-1)) \leq |U_t| < \epsilon_1 n\}] \\ &\leq \exp\left(-\frac{\eta}{2} \frac{n^b}{r-1} \log \frac{n(r-1)}{n^b}\right). \end{aligned} \quad (2.12)$$

Combining these two bounds of (2.11) and (2.12) we get

$$\begin{aligned} \mathbf{P}(F_{t+1}^c \cap H_t \cap \{t < \nu\}) &\leq \mathbf{P}((B_t \cap C_t)^c \cap H_t \cap \{t < \nu\}) \\ &\leq \mathbf{P}(B_t^c \cap H_t) + \mathbf{P}(C_t^c \cap B_t \cap H_t \cap \{t < \nu\}) \leq \exp(-n^{b/2}) \end{aligned}$$

for large n . Since $\nu \leq \lceil \epsilon_1 n - n^b \rceil$ on $H_{\lceil \epsilon_1 n - n^b \rceil}$,

$$\begin{aligned} \mathbf{P}(\nu > \lceil \epsilon_1 n - n^b \rceil) &\leq \mathbf{P}[(\nu > \lceil \epsilon_1 n - n^b \rceil) \cap (\cup_{t=1}^{\lceil \epsilon_1 n - n^b \rceil} F_t^c)] \\ &\leq \sum_{t=1}^{\lceil \epsilon_1 n - n^b \rceil} \mathbf{P}(F_t^c \cap H_{t-1} \cap \{\nu > t-1\}) \\ &\leq (\lceil \epsilon_1 n - n^b \rceil) \exp(-n^{b/2}) \leq \exp(-n^{b/4}) \end{aligned}$$

for large n and we get the result. \blacksquare

The next result shows that if there are $\lceil \epsilon n \rceil$ many occupied sites at some time for some $\epsilon > 0$, then the dual process survives for at least $\exp(cn)$ units of time for some constant c .

Lemma 2.4. *If $q(r-1) > 1$, then there exist constants $c > 0$ and $\epsilon_1 > 0$ as in Lemma 2.3 such that for $T = \exp(cn)$ and any A with $|A| \geq \lceil \epsilon_1 n \rceil$,*

$$\mathbf{P}(\inf_{t \leq T} |\hat{\xi}_t^A| < \epsilon_1 n) \leq 2 \exp(-cn).$$

Proof. Choose $\eta > 0$ so that $(q-\eta)(r-1-\eta) > 1$, and then choose $\epsilon_0(\eta) > 0$ as in Theorem 2. Take $\epsilon_1 = \epsilon_0(\eta)$. For any A with $|A| \geq \lceil \epsilon_1 n \rceil$, let $U'_t = \{x \in \hat{\xi}_t^A : x \text{ gives birth}\}$,

$t = 0, 1, \dots$. If $|U'_t| \leq \lfloor \epsilon_1 n \rfloor$, then take $U_t = U'_t$. If $|U'_t| > \epsilon_1 n$, we have too many vertices to use Theorem 2, so we let U_t be the subset of U'_t consisting of the $\lfloor \epsilon_1 n \rfloor$ vertices with smallest indices. Let

$$\begin{aligned} F_t &= \{|\hat{\xi}_t^A| \geq \lceil \epsilon_1 n \rceil\}, & H_t &= \cap_{s=0}^t F_s, \\ B_t &= \{\text{at least } (q - \eta)|\hat{\xi}_t^A| \text{ many occupied sites of } \hat{\xi}_t^A \text{ give birth}\}, \\ C_t &= \{|U_t^*| \geq (r - 1 - \eta)|U_t|\}. \end{aligned}$$

Now using an argument similar for the one for (2.9), $F_{t+1} \cap H_t \supset B_t \cap C_t \cap H_t$ for any $t \geq 0$. Using our binomial large deviations result (2.10) again, $P_{G_n}(B_t | \hat{\xi}_t^A) \geq 1 - \exp(-\Gamma((q - \eta)/q)q|\hat{\xi}_t^A|)$. On the event F_t , $|\hat{\xi}_t^A| \geq \lceil \epsilon_1 n \rceil$, and so

$$P_{G_n}(B_t^c \cap H_t) \leq P_{G_n}(B_t^c \cap \{|\hat{\xi}_t^A| \geq \lceil \epsilon_1 n \rceil\}) \leq \exp(-\Gamma((q - \eta)/q)q\epsilon_1 n).$$

The same bound works for the unconditional probability distribution \mathbf{P} .

Since $|U_t| \leq \epsilon_1 n$, and on the event $H_t \cap B_t$ $|U_t| \geq (q - \eta)\epsilon_1 n \geq \epsilon_1 n/(r - 1)$, using Theorem 2 and similar argument which leads to (2.12) we have

$$\mathbf{P}(C_t^c \cap H_t \cap B_t) \leq \exp\left(-\frac{\eta}{2} \frac{\epsilon_1 n}{r - 1} \log \frac{r - 1}{\epsilon_1}\right).$$

Combining these two bounds

$$\begin{aligned} \mathbf{P}(F_{t+1}^c \cap H_t) &\leq \mathbf{P}[(B_t \cap C_t)^c \cap H_t] \\ &\leq \mathbf{P}(B_t^c \cap H_t) + \mathbf{P}(C_t^c \cap B_t \cap H_t) \leq 2 \exp(-2c(\eta)n), \end{aligned}$$

where

$$c(\eta) = \frac{1}{2} \min \left\{ \Gamma\left(\frac{q - \eta}{q}\right) q\epsilon_1, \frac{\eta}{2} \frac{\epsilon_1}{r - 1} \log \frac{r - 1}{\epsilon_1} \right\}.$$

Hence for $T \equiv \exp(c(\eta)n)$

$$\begin{aligned} \mathbf{P}\left(\inf_{t \leq T} |\hat{\xi}_t^A| < \epsilon_1 n\right) &\leq \mathbf{P}\left(\cup_{t=1}^{\lfloor T \rfloor} F_t^c\right) \\ &\leq \sum_{t=0}^{\lfloor T \rfloor - 1} \mathbf{P}(F_{t+1}^c \cap G_t) \leq 2T \exp(-2c(\eta)n) = 2 \exp(-c(\eta)n). \end{aligned}$$

which completes the proof. \blacksquare

Lemma 2.4 confirms prolonged persistence for the dual. We will now give the

Proof of Theorem 1. Choose $\delta \in (0, qr - 1)$ and $\gamma = (20 \log r)^{-1}$. Define the random variables $Y_x, 1 \leq x \leq n$, so that $Y_x = 1$ if the dual process $\hat{\xi}^{(x)}$ starting at x satisfies $|\hat{\xi}_{\lceil 2\gamma \log n \rceil}^{(x)}| \geq \lceil n^b \rceil$ for $b = \gamma \log(qr - \delta)$, and $Y_x = 0$ otherwise. By Lemma 2.2, if n is large, then

$$\mathbf{E}Y_x \geq \rho - \delta \quad \text{for any } x.$$

Let π_x^1 , $\pi_{x,z}$ and $\alpha_x^{n,3/10}$ be the stopping times as in (2.3), and $I_x^1, I_{x,z}$ be the corresponding events as in Lemma 2.1. Recall that $\hat{G}_{x,M}$ is the subgraph with vertex set

$V_n \cap \{u : d_0(x, u) \leq M\}$. On the event $I_{x,z}$, $\hat{G}_{x, \lceil 2\gamma \log n \rceil}$ and $\hat{G}_{z, \lceil 2\gamma \log n \rceil}$ are oriented finite r -trees consisting of disjoint sets of vertices, since $2r^{\lceil 2\gamma \log n \rceil} \leq n^{1/5}$ by the choice of γ . Hence if $P_{I_{x,z}}$ is the conditional distribution of $(\hat{\xi}^{[x]}, \hat{\xi}^{[z]})$ given $I_{x,z}$, then

$$\begin{aligned} P_{I_{x,z}}[(\hat{\xi}_t^{[x]}, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot, (\hat{\xi}_t^{[z]}, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot] \\ = P_{I_{x,z}}[(\hat{\xi}_t^{[x]}, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot] P_{I_{x,z}}[(\hat{\xi}_t^{[z]}, 0 \leq t \leq \lceil 2\gamma \log n \rceil) \in \cdot]. \end{aligned}$$

Having all the ingredients ready we will now estimate the covariance between the events $\{Y_x = 1\}$ and $\{Y_z = 1\}$ for $x \neq z$. Standard probability arguments give the inequalities

$$\begin{aligned} \mathbf{P}(Y_x = 1, Y_z = 1) &\leq \mathbf{P}[(Y_x = 1, Y_z = 1) \cap I_{x,z}] + \mathbb{P}(I_{x,z}^c) \\ &= P_{I_{x,z}}(Y_x = 1, Y_z = 1) \mathbb{P}(I_{x,z}) + \mathbb{P}(I_{x,z}^c) \\ &= P_{I_{x,z}}(Y_x = 1) P_{I_{x,z}}(Y_z = 1) \mathbb{P}(I_{x,z}) + \mathbb{P}(I_{x,z}^c) \\ &= \mathbf{P}[(Y_x = 1) \cap I_{x,z}] \mathbf{P}[(Y_z = 1) \cap I_{x,z}] / \mathbb{P}(I_{x,z}) + \mathbb{P}(I_{x,z}^c) \\ &\leq \mathbf{P}(Y_x = 1) \mathbf{P}(Y_z = 1) / \mathbb{P}(I_{x,z}) + \mathbb{P}(I_{x,z}^c). \end{aligned}$$

Subtracting $\mathbf{P}(Y_x = 1) \mathbf{P}(Y_z = 1)$ from both sides gives

$$\begin{aligned} \mathbf{P}(Y_x = 1, Y_z = 1) - \mathbf{P}(Y_x = 1) \mathbf{P}(Y_z = 1) \\ \leq \mathbf{P}(Y_x = 1) \mathbf{P}(Y_z = 1) \left(\frac{1}{\mathbb{P}(I_{x,z})} - 1 \right) + \mathbb{P}(I_{x,z}^c) \\ \leq \mathbb{P}(I_{x,z}^c) [1 + 1/\mathbb{P}(I_{x,z})], \end{aligned} \tag{2.13}$$

where in the last inequality we replaced the two probabilities by 1. Now from Lemma 2.1 $\mathbb{P}(I_{x,z}^c) \leq 5n^{-3/5}$, and so

$$\mathbf{P}(Y_x = 1, Y_z = 1) - \mathbf{P}(Y_x = 1) \mathbf{P}(Y_z = 1) \leq 5n^{-3/5} (1 + 1/(1 - 5n^{-3/5})) \leq 15n^{-3/5}$$

for large enough n . Using this bound,

$$\text{var} \left(\sum_{x=1}^n Y_x \right) \leq n + 15n(n-1)n^{-3/5},$$

and Chebyshev's inequality shows that as $n \rightarrow \infty$

$$\mathbf{P} \left(\left| \sum_{x=1}^n (Y_x - \mathbf{E}Y_x) \right| \geq n\delta \right) \leq \frac{n + 15n(n-1)n^{-3/5}}{n^2\delta^2} \rightarrow 0.$$

Since $\mathbf{E}Y_x \geq \rho - \delta$, this implies

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\sum_{x=1}^n Y_x \geq n(\rho - 2\delta) \right) = 1. \tag{2.14}$$

Our next goal is to show that ξ_T^1 contains the random set $D \equiv \{x : Y_x = 1\}$ at $T = T_1 + T_2$, a time that grows exponentially fast in n . We choose $\eta > 0$ so that $(q - \eta)(r - 1 - \eta) > 1$. Let ϵ_1 and $c(\eta)$ be the constants in Lemma 2.4. If $Y_x = 1$, then $|\hat{\xi}_{T_1}^{[x]}| \geq \lceil n^b \rceil$ for

$T_1 = \lceil 2\gamma \log n \rceil$. Combining the error probabilities of Lemmas 2.3 and 2.4 shows that for $T_2 = \lfloor \exp(c(\eta)n) \rfloor + \lceil \epsilon_1 n - n^b \rceil$, and for any subset A of vertices with $|A| \geq \lceil n^b \rceil$

$$\mathbf{P}(|\hat{\xi}_{T_2}^A| \geq \lceil \epsilon_1 n \rceil) \geq 1 - 3 \exp(-n^{b/4}) \quad (2.15)$$

for large n .

Let \mathcal{C} be the set of all subsets of V_n of size at least $\lceil n^b \rceil$, and denote $C_x \equiv \hat{\xi}_{T_1}^{\{x\}}$. Using the duality relationship of (1.4) for the conditional probability distribution

$$\mathcal{P}(\cdot) = \mathbf{P}(\cdot \mid \hat{\xi}_t^{\{x\}}, 0 \leq t \leq T_1, x \in V_n),$$

we see that

$$\begin{aligned} \mathcal{P}(\xi_{T_1+T_2}^1 \supseteq D) &= \mathcal{P}[\cap_{x \in D} \{x \in \xi_{T_1+T_2}^1\}] \\ &= \mathcal{P}[\cap_{x \in D} \{\hat{\xi}_{T_1+T_2}^{\{x\}} \neq \emptyset\}]. \end{aligned}$$

Since $D = \{x : Y_x = 1\}$, it follows from the definition of Y_x that $C_x \in \mathcal{C}$ for all $x \in D$. So by the Markov property of the dual process the above is

$$\begin{aligned} &= \sum_{C_x \in \mathcal{C}, x \in D} \mathcal{P}[\cap_{x \in D} (\hat{\xi}_{T_1+T_2}^{\{x\}} \neq \emptyset, \hat{\xi}_{T_1}^{\{x\}} = C_x)] \\ &= \sum_{C_x \in \mathcal{C}, x \in D} \mathbf{P}[\cap_{x \in D} (\hat{\xi}_{T_2}^{C_x} \neq \emptyset)] \mathcal{P}[\cap_{x \in D} (\hat{\xi}_{T_1}^{\{x\}} = C_x)]. \end{aligned}$$

Using (2.15) $\mathbf{P}(\hat{\xi}_{T_2}^{C_x} \neq \emptyset) \geq \mathbf{P}(|\hat{\xi}_{T_2}^{C_x}| \geq \lceil \epsilon_1 n \rceil) \geq 1 - 3 \exp(-n^{b/4})$. So the above is

$$\begin{aligned} &\geq (1 - 3|D| \exp(-n^{b/4})) \sum_{C_x \in \mathcal{C}, x \in D} \mathcal{P}[\cap_{x \in D} (\hat{\xi}_{T_1}^{\{x\}} = C_x)] \\ &\geq 1 - 3n \exp(-n^{b/4}). \end{aligned}$$

For the last inequality we use $|D| \leq n$ and $\mathcal{P}(Y_x = 1 \forall x \in D) = 1$. Since the lower bound only depends on n , the unconditional probability

$$\mathbf{P}(\xi_{T_1+T_2}^1 \supseteq \{x : Y_x = 1\}) \geq 1 - 3n \exp(-n^{b/4}).$$

Hence for $T = T_1 + T_2$ using the attractiveness property of the threshold contact process, and combining the last calculation with (2.14) we conclude that as $n \rightarrow \infty$

$$\begin{aligned} \inf_{t \leq T} \mathbf{P}\left(\frac{|\xi_t^1|}{n} > \rho - 2\delta\right) &= \mathbf{P}\left(\frac{|\xi_T^1|}{n} > \rho - 2\delta\right) \\ &\geq \mathbf{P}\left(\xi_T^1 \supseteq \{x : Y_x = 1\}, \sum_{x=1}^n Y_x \geq n(\rho - 2\delta)\right) \rightarrow 1. \end{aligned}$$

This completes the proof of Theorem 1. ■

3. PROOF OF THEOREM 3

Recall the definition of the active sets $A_x^k, k = 0, 1, \dots, \beta_x$, and the removed sets $R_x^k, k = 0, 1, \dots, \beta_x$, introduced before Lemma 2.1. Also recall the stopping times π_x^1 and $\alpha_x^{n,\delta}$ in (2.3) and define

$$\pi_x^2 \equiv \min \{l > \pi_x^1 : |R_x^l| < l - 1\}.$$

This is the time of second collision while exploring \hat{G}_n starting from x . First we show that with high probability for every vertex $x \in V_n$ the second collision occurs after $\lceil n^{1/4-\delta} \rceil$ many steps for any $\delta \in (0, 1/4)$.

Lemma 3.1. *Let $\delta \in (0, 1/4)$ and I_x^2 be the event*

$$I_x^2 \equiv \{\pi_x^2 \wedge \beta_x \geq \alpha_x^{n,1/4+\delta}\}.$$

Then for $I \equiv \bigcap_{x \in V_n} I_x^2$, $\mathbb{P}(I^c) \leq 2n^{-4\delta}$ for large enough n .

Proof. Let $\delta' = (1/4) - \delta$. Since in the construction of the random graph G_n the input nodes $y_i(z), 1 \leq i \leq r$, for any vertex z are distinct and different from z , there are at least $n - r$ choices for each $y_i(z)$. Also $|R_x^l| \leq l$ for any l . So $\mathbb{P}(|R_x^k| = |R_x^{k-1}|) \leq (k-1)/(n-r)$. Now if I_x^2 fails to occur, then there will be k_1 and k_2 such that $1 \leq k_1 < k_2 \leq \lceil n^{\delta'} \rceil$ and $|R_x^{k_i}| = |R_x^{k_i-1}|$ for $i = 1, 2$. So

$$\begin{aligned} \mathbb{P}[(I_x^2)^c] &\leq \sum_{1 \leq k_1 < k_2 \leq \lceil n^{\delta'} \rceil} \mathbb{P}(|R_x^{k_1}| = |R_x^{k_1-1}|, |R_x^{k_2}| = |R_x^{k_2-1}|) \\ &\leq \sum_{1 \leq k_1 < k_2 \leq \lceil n^{\delta'} \rceil} \frac{(k_1-1)(k_2-1)}{(n-r)^2} \leq \sum_{1 \leq k_1, k_2 \leq \lceil n^{\delta'} \rceil} 2 \frac{(k_1-1)(k_2-1)}{n^2} \leq 2n^{4\delta'-2} \end{aligned}$$

for large enough n . The second inequality holds because the choices of the input nodes are independent. Hence $\mathbb{P}(I^c) \leq \sum_{x \in V_n} \mathbb{P}[(I_x^2)^c] \leq 2n^{4\delta'-1} = 2n^{-4\delta}$. ■

Lemma 3.1 shows that with high probability for all vertices there will be at most one collision until we have explored $\lceil n^{1/4-\delta} \rceil$ many vertices starting from any vertex of \hat{G}_n . Now recall the definition of the distance functions d_0 and d from (1.6), and $m(A, K)$ given in (1.7). Let $R = \log n / \log r$, $a = (1/8 - \delta)$ and let ρ be the branching process survival probability defined in (1.2).

Lemma 3.2. *Let P_I denote the conditional distribution of $\hat{\xi}^{(x)}$, $x \in V_n$ given I , where I is the event defined in Lemma 3.1. If $qr > 1$ and δ_0 is small enough, then for any $0 < \delta < \delta_0$ there are constants $C(\delta) > 0$, $B(\delta) = (1/8 - 2\delta) \log(qr - \delta) / \log r$ and a stopping time T satisfying*

$$P_I(T < 2 \exp(C(\delta)n^{B(\delta)})) \leq 2 \exp[-C(\delta)n^{B(\delta)}],$$

such that for any A with $m(A, 2\lceil aR \rceil) \geq \lfloor n^{B(\delta)} \rfloor$, $|\hat{\xi}_T^A| \geq \lfloor n^{B(\delta)} \rfloor$.

Proof. Let $m_t \equiv m(\hat{\xi}_t^A, 2\lceil aR \rceil)$. We define the stopping times σ_i and τ_i as follows. $\sigma_0 \equiv 0$, and for $i \geq 0$

$$\begin{aligned}\tau_{i+1} &\equiv \min\{t > \sigma_i : m_t < \lfloor n^B \rfloor\}, \\ \sigma_{i+1} &\equiv \min\{t > \tau_{i+1} : m_t \geq \lfloor n^B \rfloor\}.\end{aligned}$$

Since $\tau_i > \sigma_{i-1}$ for $i \geq 1$, $m_{\tau_{i-1}} \geq \lfloor n^B \rfloor$, and hence there is a set $X_i \subset \hat{\xi}_{\tau_{i-1}}^A$ of size at least $\lfloor n^B \rfloor$ such that $d(u, v) \geq 2\lceil aR \rceil$ for any two distinct vertices $u, v \in X_i$. Let E_i be the event that at least $(q - \delta)|X_i|$ many vertices of X_i give birth at time τ_i . Using the binomial large deviation estimate (2.10)

$$P_{G_n}(E_i) \geq 1 - \exp(-\Gamma((q - \delta)/q)q\lfloor n^B \rfloor), \quad (3.1)$$

where $\Gamma(x) = x \log x - x + 1$.

Now let I be the event defined in Lemma 3.1. Since $|\{z : d_0(x, z) \leq 2\lceil aR \rceil\}|$ is at most $2r^{2\lceil aR \rceil} \leq 2rn^{2a} \leq n^{1/4-\delta}$, so if I occurs, then for any vertex $x \in V_n$ there is at most one collision in $\{z : d_0(x, z) \leq 2\lceil aR \rceil\}$, and hence there are at least $r - 1$ input nodes $u_1(x), \dots, u_{r-1}(x)$ of x such that $\{z : d_0(u_i(x), z) \leq 2\lceil aR \rceil - 1\}$ is a finite oriented r -tree for each $1 \leq i \leq r - 1$. Since the right hand side of (3.1) depends only on n ,

$$P_I(I \cap E_i) = P_I(E_i) \geq 1 - \exp(-c_1(\delta)n^B),$$

where $c_1(\delta) = \Gamma((q - \delta)/q)q/2$. If $I \cap E_i$ occurs, then we can choose one suitable offspring of each of the vertices in X_i , which give birth, to form a subset $N_i \subset \hat{\xi}_{\tau_i}^A$ such that $|N_i| \geq (q - \delta)\lfloor n^B \rfloor$, $d(u, v) \geq 2\lceil aR \rceil - 2$ for any two distinct vertices $u, v \in N_i$, and $\{z : d_0(u, z) \leq 2\lceil aR \rceil - 1\}$ is a finite oriented r -tree for each $u \in N_i$.

By the definition of N_i it is easy to see that for each $x \in N_i$

$$P_I(|\hat{\xi}_t^{\{x\}}|, 0 \leq t \leq 2\lceil aR \rceil - 1) \in \cdot = P_{Z^x}[(Z_t^x, 0 \leq t \leq 2\lceil aR \rceil - 1) \in \cdot],$$

where Z^x is a supercritical branching process, as introduced in Lemma 2.2, with distribution P_{Z^x} and mean offspring number qr . Let B_x be the event of survival for Z^x , and $F_x = \cap_{t=\lceil \delta R \rceil-1}^{\lceil aR \rceil-2} \{Z_{t+1}^x \geq (qr - \delta)Z_t^x\}$. So $P_{Z^x}(B_x) = \rho > 0$ as in (1.2). Using the error probability of (2.7)

$$P_{Z^x}(F_x^c | B_x) \leq \sum_{t=\lceil \delta R \rceil-1}^{\lceil aR \rceil-2} e^{-c'(\delta)t} \leq C_\delta e^{-c'(\delta)\delta \log n / (2 \log r)} = C_\delta n^{-c'(\delta)\delta / (2 \log r)} \quad (3.2)$$

for some constants $C_\delta, c'(\delta) > 0$. On the event $B_x \cap F_x$,

$$Z_{\lceil aR \rceil-1}^x \geq (qr - \delta)^{(\lceil aR \rceil-1)-(\lceil \delta R \rceil-1)} \geq (qr - \delta)^{(a-\delta)R} = n^{(a-\delta) \log(qr-\delta) / \log r} = n^B.$$

Hence for $Q_x \equiv \{|\hat{\xi}_{\lceil aR \rceil-1}^{\{x\}}| \geq \lceil n^B \rceil\}$ for $x \in N_i$, we use standard probability arguments and (3.2) to have

$$\begin{aligned}P_I(Q_x) &= P_I(|\hat{\xi}_{\lceil aR \rceil-1}^{\{x\}}| \geq \lceil n^B \rceil) = P_{Z^x}(Z_{\lceil aR \rceil-1}^x \geq \lceil n^B \rceil) \\ &\geq P_{Z^x}(B_x \cap F_x) \geq P_{Z^x}(B_x)P_{Z^x}(F_x | B_x) \geq \rho - \delta\end{aligned} \quad (3.3)$$

for large enough n .

Since $d(u, v) \geq 2\lceil aR \rceil - 2$ for any two distinct vertices $u, v \in N_i$, $\hat{\xi}_i^{N_i}$ is a disjoint union of $\hat{\xi}_i^{(x)}$ over $x \in N_i$ for $t \leq \lceil aR \rceil - 1$. Let H_i be the event that there is at least one $x \in N_i$ for which Q_x occurs. Then recalling that $|N_i| \geq (q - \delta)\lfloor n^B \rfloor$ on E_i ,

$$P_I(H_i^c | E_i) \leq (1 - \rho + \delta)^{(q-\delta)\lfloor n^B \rfloor} = \exp(-c_2(\delta)n^B), \quad (3.4)$$

where $c_2(\delta) = (q - \delta) \log(1/(1 - \rho + \delta))/2$.

If $H_i \cap E_i$ occurs, choose any vertex $w_i \in N_i$ such that Q_{w_i} occurs and let $S_i \equiv \hat{\xi}_{\lceil aR \rceil - 1}^{\{w_i\}}$. By the choice of w_i , $|S_i| \geq \lfloor n^B \rfloor$. Since $(\lceil aR \rceil - 1) + \lceil aR \rceil = 2\lceil aR \rceil - 1$, for any two distinct vertices $x, z \in S_i$ the subgraphs induced by $\{u : d_0(x, u) \leq \lceil aR \rceil\}$ and $\{u : d_0(z, u) \leq \lceil aR \rceil\}$ are finite r -trees consisting of disjoint sets of vertices, and hence $d(x, z) \geq 2\lceil aR \rceil$. Hence using monotonicity of the dual process $\sigma_i \leq \tau_i + \lceil aR \rceil - 1$ on this event $H_i \cap E_i$. So

$$P_I(\sigma_i > \tau_i + \lceil aR \rceil - 1) \leq P_I(E_i^c) + P_I(H_i^c | E_i) \leq 2\exp(-2C(\delta)n^B),$$

where $C(\delta) \equiv \min\{c_1(\delta), c_2(\delta)\}/2$. Let $L = \inf\{i \geq 1 : \sigma_i > \tau_i + \lceil aR \rceil - 1\}$. Then

$$\begin{aligned} P_I[L > \exp(C(\delta)n^B)] &\geq [1 - 2\exp(-2C(\delta)n^B)]^{\exp(C(\delta)n^B)} \\ &\geq 1 - 2\exp(-C(\delta)n^B). \end{aligned}$$

Since $\sigma_i > \tau_i > \sigma_{i-1}$, $\sigma_{L-1} \geq 2(L-1)$. As $|\hat{\xi}_{\sigma_{L-1}}^A| \geq \lfloor n^B \rfloor$, we get our result if we take $T = \sigma_{L-1}$. ■

As in the proof of Theorem 1, survival of the dual process gives persistence of the threshold contact process.

Proof of Theorem 3. Let $0 < \delta < \delta_0$, ρ , $a = (1/8 - \delta)$ and $B = (1/8 - 2\delta) \log(qr - \delta) / \log r$ be the constants from the previous proof. Define the random variables Y_x , $1 \leq x \leq n$, as $Y_x = 1$ if the dual process $\hat{\xi}^{(x)}$ starting at x satisfies $|\hat{\xi}_{\lceil aR \rceil - 1}^{(x)}| > \lfloor n^B \rfloor$ and $Y_x = 0$ otherwise.

Consider the event $I_x^1 = \{\pi_x^1 \wedge \beta_x \geq \alpha_x^{n, 1/4+\delta}\}$, where π_x^1 , β_x and $\alpha_x^{n, 1/4+\delta}$ are stopping times defined as in (2.3). Using Lemma 2.1 and 3.1

$$P_I[(I_x^1)^c] \leq \frac{\mathbb{P}[(I_x^1)^c]}{\mathbb{P}(I)} \leq \frac{n^{-2(1/4+\delta)}}{1 - 2n^{-4\delta}} \leq 2n^{-(1/2+2\delta)}. \quad (3.5)$$

Let $J_x \equiv I \cap I_x^1$ and P_{J_x} be the conditional distribution of $\hat{\xi}^{(x)}$ given J_x . Since the number of vertices in the set $\{u : d_0(x, u) \leq \lceil aR \rceil - 1\}$ is at most $2r^{\lceil aR \rceil - 1} \leq 2r^{aR} < n^{1/4-\delta}$ by the choice of a ,

$$P_{J_x}[(|\hat{\xi}_t^{(x)}|, 0 \leq t \leq \lceil aR \rceil - 1) \in \cdot] = P_{Z^x}[(Z_t^x, 0 \leq t \leq \lceil aR \rceil - 1) \in \cdot],$$

where Z^x is a supercritical branching process, as introduced in Lemma 2.2, with distribution P_{Z^x} and mean offspring number qr . Let B_x and $F_x = \cap_{t=\lceil \delta R \rceil - 2}^{\lceil aR \rceil - 2} \{Z_{t+1}^x \geq (qr - \delta)Z_t^x\}$. So $P_{Z^x}(B_x) = \rho > 0$ as in (1.2), and similar to (3.2)

$$P_{Z^x}(F_x^c | B_x) \leq \sum_{t=\lceil \delta R \rceil - 2}^{\lceil aR \rceil - 2} e^{-c'(\delta)t} \leq C_\delta n^{-c'(\delta)\delta/(2 \log r)}$$

for some constants $C_\delta, c'(\delta) > 0$. On the event $B_x \cap F_x, Z_{\lceil aR \rceil - 1}^x \geq (qr - \delta)^{(\lceil aR \rceil - 1) - (\lfloor \delta R \rfloor - 2)} > (qr - \delta)^{(a-\delta)R} \geq \lfloor n^B \rfloor$. Hence using (3.5)

$$\begin{aligned} P_I(Y_x = 1) &\geq P_I(I_x^1 \cap \{|\hat{\xi}_{\lceil aR \rceil - 1}^{[x]}| > \lfloor n^B \rfloor\}) \\ &= P_{J_x}(|\hat{\xi}_{\lceil aR \rceil - 1}^{[x]}| > \lfloor n^B \rfloor) P_I(I_x^1) \\ &= P_{Z^x}(Z_{\lceil aR \rceil - 1}^x > \lfloor n^B \rfloor) P_I(I_x^1) \\ &\geq P_{Z^x}(B_x \cap F_x) P_I(I_x^1) = P_{Z^x}(B_x) P_{Z^x}(F_x | B_x) P_I(I_x^1) \geq \rho - \delta \end{aligned}$$

for large enough n .

Next we estimate the covariance between the events $\{Y_x = 1\}$ and $\{Y_z = 1\}$. We consider the stopping times $\pi_x^1, \beta_x, \pi_{x,z}, \alpha_x^{n, 1/4+\delta}$ as in (2.3) and the corresponding event $I_{x,z}$ as in Lemma 2.1. We can use similar argument, which leads to (2.13), to conclude

$$P_I(Y_x = 1, Y_z = 1) - P_I(Y_x = 1)P_I(Y_z = 1) \leq P_I(I_{x,z}^c)(1 + 1/P_I(I_{x,z})).$$

From Lemma 2.1 and 3.1,

$$P_I(I_{x,z}^c) \leq \frac{\mathbb{P}(I_{x,z}^c)}{\mathbb{P}(I)} \leq \frac{5n^{-2(1/4+\delta)}}{1 - 2N^{-4\delta}} \leq 10n^{-(1/2+2\delta)}$$

for large enough n , and so

$$P_I(Y_x = 1, Y_z = 1) - P_I(Y_x = 1)P_I(Y_z = 1) \leq 30n^{-(1/2+2\delta)}$$

for large n . Using the bound on the covariances,

$$\text{var}_I\left(\sum_{x=1}^n Y_x\right) \leq n + 30n(n-1)n^{-2\delta},$$

and Chebyshev's inequality gives that as $n \rightarrow \infty$

$$P_I\left(\left|\sum_{x=1}^n (Y_x - \mathbf{E}Y_x)\right| \geq n\delta\right) \leq \frac{n + 30n(n-1)n^{-2\delta}}{n^2\delta^2} \rightarrow 0.$$

Since $\mathbf{E}Y_x \geq \rho - \delta$ for all $x \in V_n$, this implies

$$\lim_{n \rightarrow \infty} P_I\left(\sum_{x=1}^n Y_x \geq n(\rho - 2\delta)\right) = 1. \quad (3.6)$$

Our next goal is to show that ξ_T^1 contains the random set $D \equiv \{x : Y_x = 1\}$ with high probability for a suitable choice of T . If $Y_x = 1$, then $|\hat{\xi}_{T_1}^{[x]}| > \lfloor n^B \rfloor$, where $T_1 = \lceil aR \rceil - 1$. Note that $\lceil aR \rceil - 1 + \lceil aR \rceil \leq 2\lceil aR \rceil$, and on the event I there can be at most one collision in $\{u : d_0(x, u) \leq 2\lceil aR \rceil\}$. Even though the first collision occurs between descendants of two vertices in $\hat{\xi}_{T_1}^{[x]}$, still we can exclude one vertex from $\hat{\xi}_{T_1}^{[x]}$ to have a set $W_x \subset \hat{\xi}_{T_1}^{[x]}$ of size at least $\lfloor n^B \rfloor$ such that for any two distinct vertices $z, w \in W_x$, the subgraphs induced by $\{u : d_0(z, u) \leq \lceil aR \rceil\}$ and $\{v : d_0(w, v) \leq \lceil aR \rceil\}$ are finite oriented r -trees consisting of disjoint sets of vertices, i.e. $d(z, w) \geq 2\lceil aR \rceil$. So if $Y_x = 1$, then $m(\hat{\xi}_{T_1}^{[x]}, 2\lceil aR \rceil) \geq \lfloor n^B \rfloor$ on the event I .

Using Lemma 3.2, after an additional $T_2 \geq 2 \exp(C(\delta)n^B)$ units of time, the dual process contains at least $\lfloor n^B \rfloor$ many occupied sites with P_I probability $\geq 1 - 2 \exp(-C(\delta)n^B)$.

Let \mathcal{F} be the set of all subsets of V_n of size $> \lfloor n^B \rfloor$, and denote $F_x \equiv \hat{\xi}_{T_1}^{[x]}$. Using the duality relationship of (1.4) for the conditional probability $\mathcal{P}_I(\cdot) \equiv \mathcal{P}(\cdot|I)$, where

$$\mathcal{P}(\cdot) = \mathbf{P}(\cdot | \hat{\xi}_t^{[x]}, 0 \leq t \leq T_1, x \in V_n),$$

we see that

$$\begin{aligned} \mathcal{P}_I(\xi_{T_1+T_2}^1 \supseteq D) &= \mathcal{P}_I[\cap_{x \in D} \{x \in \xi_{T_1+T_2}^1\}] \\ &= \mathcal{P}_I[\cap_{x \in D} \{\hat{\xi}_{T_1+T_2}^{[x]} \neq \emptyset\}]. \end{aligned}$$

Since $D = \{x : Y_x = 1\}$, $F_x \in \mathcal{F}$ for all $x \in D$. So by the Markov property of the dual process the above is

$$\begin{aligned} &= \sum_{F_x \in \mathcal{F}, x \in D} \mathcal{P}_I[\cap_{x \in D} (\hat{\xi}_{T_1+T_2}^{[x]} \neq \emptyset, \hat{\xi}_{T_1}^{[x]} = F_x)] \\ &= \sum_{F_x \in \mathcal{F}, x \in D} P_I[\cap_{x \in D} (\hat{\xi}_{T_2}^{F_x} \neq \emptyset)] \mathcal{P}_I[\cap_{x \in D} (\hat{\xi}_{T_1}^{[x]} = F_x)]. \end{aligned}$$

Now since $W_x \subset F_x$, using monotonicity of the dual process, $P_I(\hat{\xi}_{T_2}^{F_x} \neq \emptyset) \geq P_I(\hat{\xi}_{T_2}^{W_x} \neq \emptyset)$. Also using Lemma 3.2, $P_I(|\hat{\xi}_{T_2}^{W_x}| \geq \lfloor n^B \rfloor) \geq 1 - 2 \exp(-C(\delta)n^B)$ for any $F_x \in \mathcal{F}$. So the above is

$$\begin{aligned} &\geq (1 - 2|D| \exp(-C(\delta)n^B)) \sum_{F_x \in \mathcal{F}, x \in D} \mathcal{P}_I[\cap_{x \in D} (\hat{\xi}_{T_1}^{[x]} = F_x)] \\ &\geq 1 - 2n \exp(-C(\delta)n^B). \end{aligned}$$

For the last inequality we use $|D| \leq n$ and $\mathcal{P}_I(Y_x = 1 \forall x \in D) = 1$. Since the lower bound only depends on n ,

$$\begin{aligned} P_I(\xi_{T_1+T_2}^1 \supseteq \{x : Y_x = 1\}) &\geq 1 - 2n \exp(-C(\delta)n^B) \\ \Rightarrow \mathbf{P}(\xi_{T_1+T_2}^1 \supseteq \{x : Y_x = 1\}) &\geq \mathbb{P}(I)[1 - 3n \exp(-C(\delta)n^B)] \rightarrow 1, \end{aligned}$$

as $n \rightarrow \infty$, since $\mathbb{P}(I) \geq 1 - 2n^{-4\delta}$ by Lemma 3.1.

Hence for $T = T_1 + T_2$ using the attractiveness property of the threshold contact process, and combining the last calculation with (3.6) we conclude that as $n \rightarrow \infty$

$$\begin{aligned} \inf_{t \leq T} \mathbf{P}\left(\frac{|\xi_t^1|}{n} > \rho - 2\delta\right) &= \mathbf{P}\left(\frac{|\xi_T^1|}{n} > \rho - 2\delta\right) \\ &\geq \mathbf{P}\left(\xi_T^1 \supseteq \{x : Y_x = 1\}, \sum_{x=1}^n Y_x \geq n(\rho - 2\delta)\right) \rightarrow 1, \end{aligned}$$

which completes the proof of Theorem 3. ■

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